A- Introduction to Commodity Forwards

Consider the formula for a forward price on a financial asset:

\[ F_{0,T} = S_0 e^{(r-\delta)T} \]

Where \( S_0 \) is the spot price of the asset, \( r \) is the continuously compounded interest rate, and \( \delta \) is the continuous dividend yield on the asset. The difference between the forward price and spot price reflects the cost and benefits of delaying payment for, and receipt of, the asset.

The set of prices for different expiration dates for a given commodity is called the forward curve or the forward strip for that date.

Two terms often used by commodity traders are contango and backwardation.

- If on a given date the forward curve is upward-sloping—i.e., forward prices more distant in time are higher—then we say the market is in contango.
- If the forward curve is downward sloping, we say the market is in backwardation.

Forward curves can have portions in backwardation and portions in contango, as does that for corn.

B- Equilibrium Pricing of Commodity Forwards

We can create a synthetic commodity by combining a forward contract with a zero-coupon bond.

Consider the following investment strategy: Enter into a long commodity forward contract at the price \( F_{0,T} \) and buy a zero-coupon bond that pays \( F_{0,T} \) at time \( T \). Since the forward contract is costless, the cost of this investment strategy at time 0 is just the cost of the bond, or

\[ \text{Time 0 cash Flow} = -e^{-rt}F_{0,T} \]

At time \( T \), the strategy pays

\[ S_T - F_{0,T} + F_{0,T} = S_T \]

Where \( S_T \) is the time \( T \) price of the commodity. This investment strategy creates a synthetic commodity, in that it has the same value as a unit of the commodity at time \( T \). Note that, from Equation 2, the cost of the synthetic commodity is the prepaid forward price, \( e^{-rt}F_{0,T} \).

Computing the time 0 value of a unit of the commodity received at time \( T \) is a standard problem: You discount the expected commodity price to determine its value today. Let \( E_0(S_T) \) denote the expected time-\( T \) price as of time 0, and let \( \alpha \) denote the appropriate discount rate for a time-\( T \) cash flow of \( S_T \). Then the present value is

\[ E_0(S_T) e^{-\alpha T} \]
Equations 2 and 3 represent the same value. Both reflect what you would pay today to receive one unit of the commodity at time T. Equating the two expressions, we have

$$e^{-rt}F_{0,T} = E_0(S_T)e^{-at}$$  \hspace{1cm} \text{Equation 4}

Rearranging this equation, we can write the forward price as

$$F_{0,T} = e^{rt}E_0(S_T)e^{-at}$$

$$F_{0,T} = E_0(S_T)e^{(r-a)t}$$  \hspace{1cm} \text{Equation 5}

As with financial forwards, the forward price is a biased estimate of the expected spot price, $E_0(S_T)$, with the bias due to the risk premium on the commodity, $\alpha - r$.

C- Nonstorability: Electricity

The forward market for electricity illustrates forward pricing when storage is not possible. Electricity is produced in different ways: from fuels such as coal and natural gas, or from nuclear power, hydroelectric power, wind power, or solar power. Once it is produced, electricity is transmitted over the power grid to end-users. Electricity has characteristics that distinguish it not only from financial assets, but from other commodities as well:

- First, electricity is difficult to store, hence it must be consumed when it is produced or else it is wasted.
- Second, at any point in time the maximum supply of electricity is fixed. You can produce less but not more.
- Third, demand for electricity varies substantially by season, by day of week, and by time of day.

Because electricity cannot be stored, its price is set by demand and supply at a point in time. Because of peak-load plants that operate only when prices are high, power suppliers are able to temporarily increase the supply of electricity. However, expectations about supply are already reflected in the forward price.

D- The Commodity Lease Rate

The commodity analogue to dividend income is lease income, which may not be directly observable.

1- The Lease Market for a Commodity

Suppose that $\alpha$ is the expected return on a stock that has the same risk as the commodity; $\alpha$ is therefore the appropriate discount rate for the cash flow $S_T$. The NPV of the investment is

$$NPV = E_0(S_T)e^{-\alpha T} - S_0$$  \hspace{1cm} \text{Equation 6}

Suppose that we expect the commodity price to increase at the rate $g$, so that
Then from Equation 6, the NPV of the commodity loan, without payments, is

\[ NPV = S_0 e^{(g - \alpha)T} - S_0 \]  

 Equation 7

If \( g < \alpha \), the commodity loan has a negative NPV. However, suppose the lender demands that the borrower return \( e^{(\alpha - g)T} \) units of the commodity for each unit borrowed. If one unit is loaned, \( e^{(\alpha - g)T} \) units will be returned. This is like a continuous proportional lease payment of \( \alpha - g \) to the lender. Thus, the lease rate is the difference between the commodity discount rate and the expected growth rate of the commodity price, or

\[ \delta_t = \alpha - g \]  

 Equation 8

With this payment, the NPV of a commodity loan is

\[ NPV = S_0 e^{(\alpha - g)T} e^{(g - \alpha)T} - S_0 = 0 \]  

 Equation 9

Now the commodity loan is a fair deal for the lender. The commodity lender must be compensated by the borrower for the opportunity cost associated with lending.

Note that if \( ST \) were the price of a nondividend-paying stock, its expected rate of appreciation would equal its expected return, so \( g = \alpha \) and no payment would be required for the stock loan to be a fair deal.

**2. Forward Prices and the Lease Rate**

The lease payment is a dividend. If we borrow the asset, we have to pay the lease rate to the lender, just as with a dividend-paying stock. If we buy the asset and lend it out, we receive the lease payment. Thus, the formula for the forward price with a lease market is:

\[ F_{0,T} = S_0 e^{(r - \delta_t)T} \]  

 Equation 10

The lease rate has to be consistent with the forward price. Thus, when we observe the forward price, we can infer what the lease rate would have to be if a lease market existed. Specifically, if the forward price is \( F_{0,T} \), the annualized lease rate is:

\[ \delta_t = r - \frac{1}{T} \ln \left( \frac{F_{0,T}}{S} \right) \]  

 Equation 11

If instead we use an effective annual interest rate, \( r \), the effective annual lease rate is

\[ \delta_T = \frac{(1 + r)}{(F_{0,T}/S)^{1/T}} - 1 \]  

 Equation 12

The denominator in this expression annualizes the forward premium.
By definition, contango—an upward-sloping forward curve—occurs when the lease rate is less than the risk-free rate. Backwardation—a downward-sloping forward curve—occurs when the lease rate exceeds the risk-free rate.

E- Carry Markets

One reason for storage is seasonal variation in either supply or demand, which causes a mismatch between the time at which a commodity is produced and the time at which it is consumed.

1- Storage Costs and Forward Prices

The cash-and-carry logic with storage costs suggests that you will store only if the present value of selling at time $T$ is at least as great as that of selling today. Denote the future value of storage costs for one unit of the commodity from time 0 to $T$ as $\lambda(0, T')$. Indifference between selling today and at time $T'$ requires
\[ S_0 = e^{-rT}[F_{0,T} - \lambda(0,T)] \]

This relationship in turn implies that if storage is to occur, the forward price is at least

\[ F_{0,T} \geq S_0 e^{rT} + \lambda(0,T) \]  

Equation 13

In the special case where storage costs are paid continuously and are proportional to the value of the commodity, storage cost is like a continuous negative dividend of \( \lambda \), and we can write the forward price as

\[ F_{0,T} = S_0 e^{(r+\lambda)T} \]  

Equation 14

The selling price must compensate the commodity merchant for both the financial cost of storage (interest) and the physical cost of storage. With storage costs, the forward curve can rise faster than the interest rate. We can view storage costs as a negative dividend in that, instead of receiving cash flow for holding the asset, you have to pay to hold the asset.

**2- Storage Costs and the Lease Rate**

If you lend the commodity, you are saved from having to pay storage cost. Thus, the **lease rate should equal the negative of the storage cost**. In other words, the lender will pay the borrower! In effect, the commodity borrower is providing “virtual storage” for the commodity lender, who receives back the commodity at a point in the future. The lender making a payment to the borrower generates a negative dividend.

**3- The Convenience Yield**

The convenience yield is the nonmonetary return offered by an asset when the asset is in short supply, often associated with assets with seasonal production processes.

Suppose the continuously compounded convenience yield is \( c \), proportional to the value of the commodity. The commodity lender saves \( \lambda - c \) by not physically storing the commodity; hence, the commodity borrower pays \( \delta = c - \lambda \), compensating the lender for convenience yield less storage cost. We then conclude that the forward price must be no less than

\[ F_{0,T} \geq S_0 e^{(r-\delta)T} = S_0 e^{(r+\lambda-c)T} \]

Now consider what happens if you perform a cash-and-carry, buying the commodity and selling it forward. **If you are an average investor, you will not earn the convenience yield (it is earned only by those with a business reason to hold the commodity).** You could try to lend the commodity, reasoning that the borrower could be a commercial user to whom you would pay storage cost less the convenience yield. But **those who earn the convenience yield likely already hold the optimal amount**
of the commodity. There may be no way for you to earn the convenience yield when performing a cash-and-carry. Those who do not earn the convenience yield will not own the commodity.

Thus, for an average investor, the cash-and-carry has the cash flows:

\[ F_{0,T} - S_T + S_T - S_0e^{(r+\lambda)T} = F_{0,T} - S_0e^{(r+\lambda)T} \]

This expression implies that the forward price must be below \( S_0e^{(r+\lambda)T} \) if there is to be no cash-and-carry arbitrage. In summary, from the perspective of an arbitrageur, the price range within which there is no arbitrage is

\[ S_0e^{(r+\lambda-c)T} \leq F_{0,T} \leq S_0e^{(r+\lambda)T} \]

The convenience yield produces a no-arbitrage region rather than a no-arbitrage price. The observed lease rate will depend upon both storage costs and convenience.

The difficulty with the convenience yield in practice is that convenience is hard to observe. The concept of the convenience yield serves two purposes.

- First, it explains patterns in storage—for example, why a commercial user might store a commodity when the average investor will not.
- Second, it provides an additional parameter to better explain the forward curve. You might object that we can invoke the convenience yield to explain any forward curve, and therefore the concept of the convenience yield is vacuous.

**F- Gold Futures**

To derive the lease rate from gold and Eurodollar Futures prices:

1. Determine LIBOR rates from Eurodollar; the convention for Eurodollar is 91 days and for the LIBOR 90 days thus the 91/90 adjustment
2. Adjust rates to the relevant period
3. Use equation 12 \( \delta_T = \frac{(1+r)}{(F_{0,T}/S)^{1/T}} - 1 \) to calculate lease rate

**1- Gold Investments**

If you wish to hold gold as part of an investment portfolio, you can do so by holding physical gold or synthetic gold—i.e., holding T-bills and going long gold futures. If you hold physical gold without lending it, and if the lease rate is positive, you forgo the lease rate. You also bear storage costs. With synthetic gold, on the other hand, you have a counterparty who may fail to pay so there is credit risk. Ignoring credit risk, however, synthetic gold is generally the preferable way to obtain gold price exposure. If the lease rate is negative then it has to compensate the buyer for storage costs.

Some nonfinancial holders of gold will obtain a convenience yield from gold. Consider an electronics manufacturer who uses gold in producing components. Stocking out would have a real financial cost, and the manufacturer is willing to pay a price—the lease rate—to avoid that cost.
**2- Evaluation of Gold Production**

Suppose we have an operating gold mine and we wish to compute the present value of future production. The present value of the commodity received in the future is simply the present value—computed at the risk-free rate—of the forward price. We can use the forward curve for gold to compute the value of an operating gold mine.

Suppose that at times $t_i, i = 1, \ldots, n$, we expect to extract $n t_i$ ounces of gold by paying an extraction cost $x(t_i)$. We have a set of $n$ forward prices, $F_{0,t_i}$. If the continuously compounded annual risk-free rate from time 0 to $t_i$ is $r(0, t_i)$, the value of the gold mine is

$$PV\ gold\ production = \sum_{i=1}^{n} n_i [F_{0,t_i} - x(t_i)] e^{-r(0,t_i)t_i}$$

Equation 16

This equation assumes that the gold mine is certain to operate the entire time and that the quantity of production is known. Only price is uncertain. Note that in Equation 16, by computing the present value of the forward price, we compute the prepaid forward price.

**G- Seasonality: The Corn Forward market**

Corn is produced at one time of the year, but consumed throughout the year. In order to be consumed when it is not being produced, corn must be stored. An equilibrium with some current selling and some storage requires that corn prices be expected to rise at the interest rate plus storage costs, which implies that there will be an upward trend in the price between harvests. Once the harvest begins, storage is no longer necessary; if supply and demand remain constant from year to year, the harvest price will be the same every year. The corn price will fall to that level at harvest, only to begin rising again after the harvest.

The forward price in never violates the no-arbitrage condition which says that the forward price from $T$ to $T + s$ cannot rise faster than interest plus storage costs:

$$F_{0,T+s} < F_{0,T}e^{rs} + \lambda(T, T + s)$$

Equation 18

**H- Natural Gas**

Natural gas has several interesting characteristics.

- First, gas is **costly to transport internationally**, so prices and forward curves vary regionally.
- Second, once a given well has begun production, gas is **costly to store**.
- Third, **demand for gas in the United States is highly seasonal**, with peak demand arising from heating in winter months.
Thus, there is a relatively steady stream of production with variable demand, which leads to large and predictable price swings.

Because of the expense in transporting gas internationally, the seasonal behavior of the forward curve can vary in different parts of the world. In tropical areas where gas is used for cooking and electricity generation, the forward curve is relatively flat because demand is relatively flat. In the Southern Hemisphere, where seasons are reversed from the Northern Hemisphere, the forward curve will peak in June and July rather than December and January.

I- Oil

Both oil and natural gas produce energy and are extracted from wells, but the different physical characteristics and uses of oil lead to a very different forward curve than that for gas. Oil is easier to transport than gas. Transportation of oil takes time, but oil has a global market. Oil is also easier to store than gas. Thus, seasonals in the price of crude oil are relatively unimportant.

J- Commodity Spreads

Some commodities are inputs in the creation of other commodities, which gives rise to commodity spreads. For example, crude oil is refined to make petroleum products, in particular heating oil and gasoline. The refining process entails distillation, which separates crude oil into different components, including gasoline, kerosene, and heating oil. The split of oil into these different components can be complemented by a process known as “cracking”; hence, the difference in price between crude oil and equivalent amounts of heating oil and gasoline is called the crack spread.

Oil can be processed in different ways, producing different mixes of outputs. The spread terminology identifies the number of gallons of oil as input, and the number of gallons of gasoline and heating oil as outputs. Traders will speak of “5-3-2,” “3-2-1,” and “2-1-1” crack spreads. The 5-3-2 spread, for example, reflects the profit from taking 5 gallons of oil as input, and producing 3 gallons of gasoline and 2 gallons of heating oil. A petroleum refiner producing gasoline and heating oil could use a futures crack spread to lock in both the cost of oil and output prices. This strategy would entail going long oil futures and short the appropriate quantities of gasoline and heating oil futures. Of course there are other inputs to production and it is possible to produce other outputs, such as jet fuel, so the crack spread is not a perfect hedge.

K- Hedging Strategies

Basis risk is when the price of the commodity underlying the futures contract moves differently than the price of the commodity you are hedging.

1- Basis Risk

Basis risk is a generic problem with commodities because of storage and transportation costs and quality differences. Basis risk can also arise with financial futures, as for example when a company hedges its own borrowing cost with the Eurodollar contract.
An individual stock can be hedged with an index futures contract. If we regress the individual stock return on the index return, the resulting regression coefficient provides a hedge ratio that minimizes the variance of the hedged position.

- We engage in a **strip hedge** when we hedge a stream of obligations by offsetting each individual obligation with a futures contract matching the maturity and quantity of the obligation. For the oil producer obligated to deliver every month at a fixed price, the hedge would entail buying the appropriate quantity each month, in effect taking a long position in the strip.
- An alternative to a strip hedge is a **stack hedge**. With a stack hedge, we enter into futures contracts with a single maturity, with the number of contracts selected so that changes in the present value of the future obligations are offset by changes in the value of this “stack” of futures contracts.

This process of stacking futures contracts in the near-term contract and rolling over into the new near-term contract is called a **stack and roll**. If the new near-term futures price is below the expiring near-term price (i.e., there is backwardation), rolling is profitable.

Why would anyone use a stack hedge? There are at least two reasons. First, there is often more trading volume and liquidity in near-term contracts. With many commodities, bid-ask spreads widen with maturity. Thus, a stack hedge may have lower transaction costs than a strip hedge. Second, the manager may wish to speculate on the shape of the forward curve.

### 2- Hedging Jet Fuel with Crude Oil

Jet fuel futures do not exist in the United States, but firms sometimes hedge jet fuel with crude oil futures along with futures for related petroleum products. In order to perform this hedge, it is necessary to understand the relationship between crude oil and jet fuel prices. If we own a quantity of jet fuel and hedge by holding $H$ crude oil futures contracts, our mark-to-market profit depends on the change in the jet fuel price and the change in the futures price:

$$(P_t - P_{t-1}) + H(F_t - F_{t-1})$$

where $P_t$ is the price of jet fuel and $F_t$ the crude oil futures price. We can estimate $H$ by regressing the change in the jet fuel price (**denominated in cents per gallon**) on the change in the crude futures price (**denominated in dollar per barrel**).

### 3- Weather Derivatives

Weather derivatives provide another illustration of cross-hedging. Firms could hedge their risk using weather derivatives—contracts that make payments based upon realized characteristics of weather—to cross-hedge their specific risk.